MATHEMATICS
B.Sc (Part-1), Paper II
Differential Calculus
Topic- Tangents and Normals

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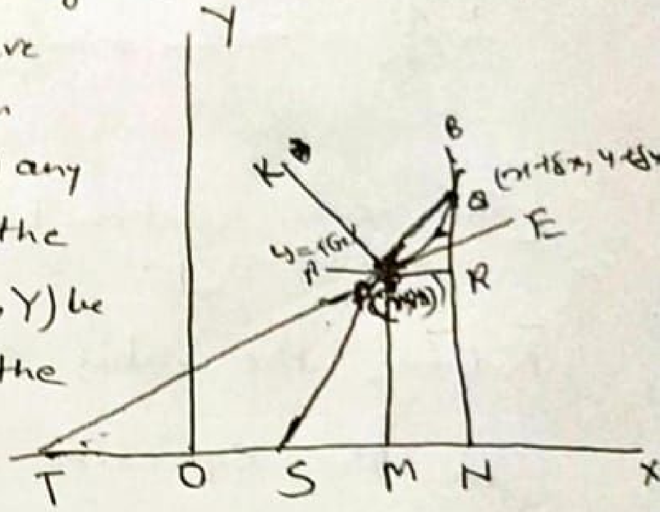
(I) To find the equation of the tangent:-



Let the equation of the curve AB is $y = f(x)$ and let the given point P on the curve be (x, y) any other neighbouring point Q on the curve be $(x + \delta x, y + \delta y)$. Let (X, Y) be the current coordinates, then the equation of the chord PQ is given by

$$\frac{Y-y}{y+\delta y-y} = \frac{X-x}{x+\delta x-x}$$

or $$\frac{Y-y}{\delta y} = \frac{X-x}{\delta x}$$

or $$(Y-y) = \frac{\delta y}{\delta x}(X-x)$$

Taking the limits as $\delta x \to 0$ we get the required equation of the tangent at $P(x, y)$ is

$$(Y-y) = \underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}(X-x) = \frac{dy}{dx}(X-x)$$

$$\therefore (Y-y) = \frac{dy}{dx}(X-x)$$

is the equation of the tangent at $P(x, y)$

II If the equation of the curve is of the form $f(x,y)=0$

We know that the partial differential equation of $f(x,y)=0$ is $\frac{\delta f}{\delta x} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dx} = 0$ [where f stands for $f(x,y)$]

$$\therefore \frac{dy}{dx} = -\frac{\frac{\delta f}{\delta x}}{\frac{\delta f}{\delta y}} = -\frac{f_x}{f_y}$$ where $f_x = \frac{\delta f}{\delta x}$

and $f_y = \frac{\delta f}{\delta y}$

[The above relation holds for all implicit relations $f(x,y)=0$].

Putting the value of $\frac{dy}{dx}$ from the equation $\frac{dy}{dx} = -\frac{f_x}{f_y}$

in the equation $(Y-y) = \frac{dy}{dx}(X-x)$.

We have $Y-y = -\frac{f_x}{f_y}(X-x)$.

or $f_x(X-x) + f_y(Y-y) = 0$

or $(X-x)\frac{\delta f}{\delta x} + (Y-y)\frac{\delta f}{\delta y} = 0$

This is the required equation of the tangent at the point P of the curve $f(x,y) = 0$.

NORMAL

(I) Find the equation of the normal to the curve $y=f(x)$ at the point $P(x,y)$.

Normal to a curve at any point is a straight line through a given point perpendicular to the tangent at that point.

Now we find equation of the normal at $P(x,y)$ to the curve $y=f(x)$

Any line passing through (x,y) is $Y-y=m(X-x)$ —(1)

where (X,Y) are the current coordinates of any point on the line.

If (1) is normal, then by definition

$$m\frac{dy}{dx}=-1 \quad \text{i.e. } m=-\frac{1}{\frac{dy}{dx}}$$

∴ The equation of the normal at (x,y) is

$$Y-y=-\frac{1}{\frac{dy}{dx}}(X-x)$$

$$\text{i.e. } (X-x)+\frac{dy}{dx}(Y-y)=0$$

(II) If the equation of the curve be $f(x,y)=0$ we know that the equation of the tangent to the curve $f(x,y)=0$ is

$$(X-x)f_x+(Y-y)f_y=0$$

$$\text{or } Y-y=-\frac{f_x}{f_y}(X-x)$$

Hence the equation of the normal to the curve $y=f(x)$ is

$$Y-y=\frac{1}{\frac{f_x}{f_y}}(X-x)$$

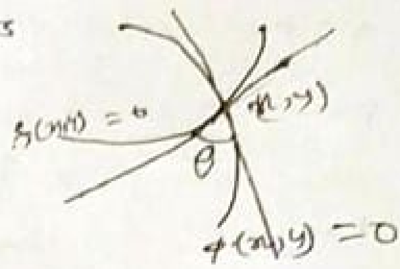
$$\text{or } (Y-y)=\frac{f_y}{f_x}(X-x)$$

$$\text{i.e. } \frac{X-x}{f_x}=\frac{Y-y}{f_y}.$$

[See the figure on page (1)]

Angle of intersection of two curves

By the angle of intersection of two curves, we mean the angle between the tangents of two curves at a common point.



Let $f(x,y)=0$ and $\phi(x,y)=0$ be the equation of two curves which intersect at the point (x,y).

The tangents to the curve at (x,y) are

$$Xf_x + Yf_y - (xf_x + yf_y) = 0$$

or $X\phi_x + Y\phi_y - (x\phi_x + y\phi_y) = 0$

Let θ be the angle between these two tangents and the angle θ is given by

$$\tan\theta = \frac{f_x\phi_y - \phi_x f_y}{f_x\phi_x + f_y\phi_y}$$

$\left[\because \tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}\right.$ where $m_1 = -\frac{f_x}{f_y}$, $m_2 = -\frac{\phi_x}{\phi_y}$

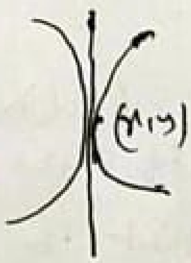
If the curves touch at (x,y) then $\theta = 0$

i.e. $\tan\theta = 0$

$$\therefore 0 = \frac{f_x\phi_y - \phi_x f_y}{f_x\phi_x + f_y\phi_y}$$

i.e. $f_x\phi_y - \phi_x f_y = 0$

i.e. $\boxed{\frac{f_x}{f_y} = \frac{\phi_x}{\phi_y}}$



and if they cut orthogonally at (x,y) then

$\theta = \frac{\pi}{2}$ i.e. $\tan\frac{\pi}{2} = \infty$

$$\therefore \infty = \frac{f_x\phi_y - \phi_x f_y}{f_x\phi_x + f_y\phi_y}$$

Note: If the equation of the curve are given in the form $y = f(x)$ and $y = \phi(x)$, the angle of their intersection is given by

$$\tan^{-1}\frac{f'(x) - \phi'(x)}{1 + f'(x)\phi'(y)}$$

$\left[\because y = \phi(x) \text{ \& } y = f(x)\right.$

$\therefore \frac{dy}{dx} = m_2 = \phi'(x)$

$\left.\frac{dy}{dx} = m_1 = f'(x)\right]$

Thus, the curves cut orthogonally if $f'(x)\phi'(x) = -1$.